F. Spin is the best playground to learn Quantum Mechanics

- QM Spin AM is the simplest system that illustrates both the physics and mathematical structure of QM
- $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_z$, $\hat{S}^2$ in matrix forms satisfy the AM commutators

(a) Eigenvalues and Eigenstates

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\hat{S}_z \overbrace{\begin{pmatrix}1\\ 0\end{pmatrix}} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \overbrace{\frac{\hbar}{2}}^{\text{eigenvalue}}\overbrace{\begin{pmatrix}1\\ 0\end{pmatrix}}^{\text{eigenstate}}$$

This can be obtained by inspection.

$$\hat{S}_z \underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}} = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \underbrace{-\frac{\hbar}{2}}_{\text{eigenvalue}}\underbrace{\begin{pmatrix}0\\ 1\end{pmatrix}}_{\text{eigenstate}}$$

eigenvalue   eigenstate (eigenvector)

$\hat{S}_z$ $+\frac{\hbar}{2}$

$$\underbrace{\left|\underset{\overset{\uparrow}{s}}{\tfrac{1}{2}}\ \underset{\overset{\uparrow}{m_s}}{\tfrac{1}{2}}\right\rangle_{\underset{\nwarrow\ \text{for } \hat{S}_{\boxed{z}}}{z}} = \chi_{\frac{1}{2}\frac{1}{2}} = |m_s=\tfrac{1}{2}\rangle = |\uparrow\rangle_z}_{\text{these are all the notations commonly found}} = \overbrace{\alpha_z \equiv \begin{pmatrix}1\\0\end{pmatrix}}$$

$\hat{S}_z$ $-\frac{\hbar}{2}$

$$\left|\tfrac{1}{2}\ \underset{\overset{\uparrow}{m_s}}{-\tfrac{1}{2}}\right\rangle = \chi_{\frac{1}{2}-\frac{1}{2}} = |m_s=-\tfrac{1}{2}\rangle = |\downarrow\rangle_z = \underbrace{\beta_z \equiv \begin{pmatrix}0\\1\end{pmatrix}}$$

How about $\hat{S}_x$?

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

Solve for eigenvalues of σ_x

$$\begin{vmatrix}-\lambda & 1\\1 & -\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$\therefore$ Eigenvalues of $\hat{S}_x$ are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

For each eigenvalue, then look for the eigenvector

E.g. $$\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x}\begin{pmatrix} a \\ b \end{pmatrix} = \underset{\substack{\uparrow \\ \text{eigenvalue "+1" of } \sigma_x}}{+1}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \left.\begin{matrix} a = b \\ b = a \end{matrix}\right\rangle \text{ same information}$$

$\therefore$ Eigenvector is $\begin{pmatrix} a \\ a \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

a fixed by normalization

<u>Normalization</u>: $$|a|^2 (1 \;\; 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2|a|^2 = 1 \Rightarrow a = \frac{1}{\sqrt{2}}$$

[this is the inner product of $\begin{pmatrix} a \\ a \end{pmatrix}$ with itself]

$\therefore$ $\hat{S}_x$: eigenvalue $+\frac{\hbar}{2}$ eigenvector

$$|\tfrac{1}{2}\;\tfrac{1}{2}\rangle_x = |\uparrow\rangle_x = \alpha_x \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

referring to $\hat{S}_x$

Do the same for eigenvalue $-\frac{\hbar}{2}$

eigenvector is $|\frac{1}{2}\ -\frac{1}{2}\rangle_x = |\downarrow\rangle_x = \beta_x \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}$

<u>How about $\hat{S}_y$?</u> [Do the same]

Eigenvalues are (again) $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

[as discussed under orbital AM, the components on <u>ANY</u> direction are the same]

$\hat{S}_y$: eigenvalue $+\frac{\hbar}{2}$ eigenvector $|\frac{1}{2}\ \frac{1}{2}\rangle_y = |\uparrow\rangle_y = \alpha_y \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$\hat{S}_y$: eigenvalue $-\frac{\hbar}{2}$ eigenvector $|\frac{1}{2}\ -\frac{1}{2}\rangle_y = |\downarrow\rangle_y = \beta_y \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

The eigenvalues and eigenvectors of the matrices representing the angular momentum components of a spin-half particle

Spin component	Eigenvalue	Eigenvector
$\hat{S}_x$	$\frac{1}{2}\hbar$	$\alpha_x \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\hat{S}_x$	$-\frac{1}{2}\hbar$	$\beta_x \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$\hat{S}_y$	$\frac{1}{2}\hbar$	$\alpha_y \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\hat{S}_y$	$-\frac{1}{2}\hbar$	$\beta_y \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$
$\hat{S}_z$	$\frac{1}{2}\hbar$	$\alpha_z \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$\hat{S}_z$	$-\frac{1}{2}\hbar$	$\beta_z \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(23)

(b) Using Spin to Practise Quantum Theory of Measurement

Recall: Measure A, outcomes are eigenvalues of $\hat{A}$

Given Ψ, $\Psi = \sum_i c_i \phi_i$; $\hat{A}\phi_i = a_i \phi_i$

$|c_i|^2$ = Prob. of getting the outcome a_i (24)

Behind the scene, measurements on identical copies. . . .

Let's practice!

A generate state of spin-½ particle can be expressed as a vector ("state vector") of the form

$$\begin{pmatrix} c \\ d \end{pmatrix} \qquad \underbrace{\text{(assumed normalized)}}_{\text{Meaning: } |c|^2+|d|^2=1} \qquad (25)$$

Q1: Passing a beam of particle all prepared in state (25) into Stern-Gerlach (SG) experiment arranged to measure S_z (z-component) (Symbol: SGZ), what will happen?

$$\begin{pmatrix} c \\ d \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (26)$$

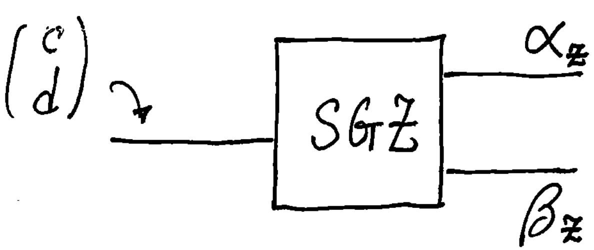
$$\left[\Psi = c\,\alpha_z + d\,\beta_z \qquad \text{c.f. } \Psi = \sum_i c_i \phi_i \text{ in (24)} \right]$$

Slow Motion: Measure S_z, look at $\underbrace{\hat{S}_z \text{ eigenvalues \& eigenstates}}_{\text{solved (see Eq. (23))}}$

Expand $\begin{pmatrix} c \\ d \end{pmatrix}$ in terms of $\hat{S}_z$'s eigenstates

From (26), $|c|^2$ is the probability of getting S_z to be $+\frac{\hbar}{2}$

$|d|^2$ is the probability of getting S_z to be $-\frac{\hbar}{2}$



α_z collapsed to $\alpha_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ probability $|c|^2$ getting $+\frac{\hbar}{2}$

β_z collapsed to $\beta_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ probability $|d|^2$ getting $-\frac{\hbar}{2}$

[This also provides a way to prepare a beam with particles all in the same spin state, e.g. take one output beam.]

Q2: Same as in Q1, but apparatus arranged to measure $\underline{S_x}$ ($\underline{SGX}$)

Expand $\begin{pmatrix} c \\ d \end{pmatrix}$ in terms of α_x and β_x

$\uparrow$ eigenstates of $\hat{S}_x$ (see Eq. (23))

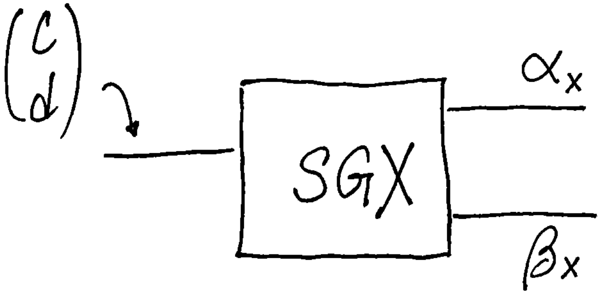
i.e.
$$\begin{pmatrix} c \\ d \end{pmatrix} = c_1 \alpha_x + c_2 \beta_x = c_1 \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\therefore c_1 = \frac{c+d}{\sqrt{2}} \quad ; \quad c_2 = -\frac{(c-d)}{\sqrt{2}}$$

$$\therefore \begin{pmatrix} c \\ d \end{pmatrix} = \frac{c+d}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{(c-d)}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad (27)$$

From (27), $\frac{|c+d|^2}{2}$ is probability of getting S_x to be $+\frac{\hbar}{2}$

$\frac{|c-d|^2}{2}$ is probability of getting S_x to be $-\frac{\hbar}{2}$



collapsed to $\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ prob. $\frac{|c+d|^2}{2}$ getting $+\frac{\hbar}{2}$

collapsed to $\beta_x = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ prob. $\frac{|c-d|^2}{2}$ getting $-\frac{\hbar}{2}$

<u>Note</u>: It is the same beam $\begin{pmatrix} c \\ d \end{pmatrix}$ to start with in Q1 & Q2, only what is being measured is different.

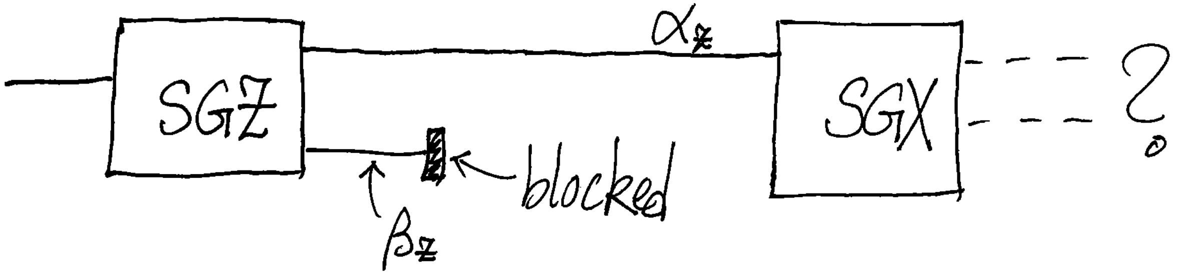
In both cases, $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ are the possible outcomes.

But the probabilities depend on whether it is SGX OR SGZ.

Q3: Same as Q1 and Q2, but now arranged to measure S_y [i.e. SGY]. (Ex.)

(c) $[\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y$ ⇒ State of definite S_z has no definite S_x, etc.

- Only when $[\hat{A}, \hat{B}] = 0$, then $\hat{A}, \hat{B}$ share common eigenstates, ∘∘∘ (Ch. X)



$\alpha_z = \begin{pmatrix}1\\0\end{pmatrix}$ with definite $S_z = \frac{\hbar}{2}$ (after SGZ, select a beam)

Send $\begin{pmatrix}1\\0\end{pmatrix}$ beam into SGX, what will happen?

[No new technique, just follow same technique]

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}_{\alpha_x} - \frac{1}{\sqrt{2}}\cdot\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}}_{\beta_x} \qquad (28)$$

↗ definite $S_z = \frac{+\hbar}{2}$

$$\therefore \begin{cases}\text{Prob. } 1/2 \text{ of getting } S_x \text{ to be } +\frac{\hbar}{2} \\ \text{Prob. } 1/2 \text{ of getting } S_x \text{ to be } -\frac{\hbar}{2}\end{cases} \begin{matrix}\searrow\\ \nearrow\end{matrix} \langle S_z \rangle = 0$$

Eq. (28) ⇛ Eigenstate of $\hat{S}_z$ is <u>NOT</u> eigenstate of $\hat{S}_x$.
($\because$ $\hat{S}_z$ and $\hat{S}_x$ do not commute)

What if we take the $S_x = +\frac{\hbar}{2}$ output beam and do SGZ again?

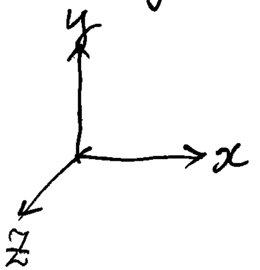
- $\alpha_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$

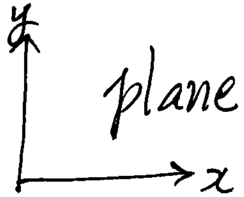
$\therefore$ No definite S_z prediction! (although started out with α_z)

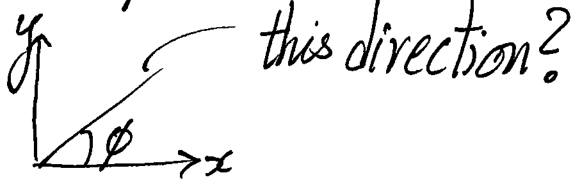
Prob. 1/2 of getting $S_z = +\frac{\hbar}{2}$, Prob. 1/2 of getting $S_z = \frac{-\hbar}{2}$

- This can go on and on.
- These QM predictions have been tested over the years!
- Experiments can also be done by photons.

(d) Nothing Special about x, y, z-directions

- Along any direction, same $+\frac{\hbar}{2}$, $-\frac{\hbar}{2}$ components
- 





$$\hat{S}_\phi = \hat{S}_x \cos\phi + \hat{S}_y \sin\phi = \frac{\hbar}{2}\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \quad (29)$$

Ex: Show eigenvalues are $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$. Find eigenstates.

Remarks

- How about summing two momenta? (more advanced course) Orbital AM($\vec{L}$) + Spin AM($\vec{S}$) "Spin-Orbit interaction"

- Measurements using SG set up [e.g. SGϕ] is important in testing QM's predictions against some alternative proposals.